Exact Solutions of Einstein's Field Equations

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We examine various well known exact solutions available in the literature to investigate the recent criterion obtained in Negi and Durgapal [*Gravitation and Cosmology* **7**, 37 (2001)] which should be fulfilled by any static and spherically symmetric solution in the state of hydrostatic equilibrium. It is seen that this criterion is fulfilled only by (i) the regular solutions having a vanishing surface density together with pressure, and (ii) the singular solutions corresponding to a non-vanishing density at the surface of the configuration. On the other hand, the regular solutions corresponding to a non-vanishing surface density do not fulfill this criterion. Based upon this investigation, we point out that the exterior Schwarzschild solution itself provides necessary conditions for the types of the density distributions to be considered inside the mass, in order to obtain exact solutions or equations of state compatible with the state of hydrostatic equilibrium in general relativity. The regular solutions with finite centre and non-zero surface densities which do not fulfill the criterion given by Negi and Durgapal (2001), in fact, cannot meet the requirement of the 'actual mass' set up by exterior Schwarzschild solution. The only regular solution which could be possible in this regard is represented by uniform (homogeneous) density distribution. This criterion provides a necessary and sufficient condition for any static and spherical configuration (including core-envelope models) to be compatible with the structure of general relativity [that is, the state of hydrostatic equilibrium in general relativity]. Thus, it may find application to construct the appropriate core-envelope models of stellar objects like neutron stars and may be used to test various equations of state for dense nuclear matter and the models of relativistic star clusters with arbitrary large central redshifts.

1. INTRODUCTION

The first two exact solutions of Einstein's field equations were obtained by Schwarzschild (Schwarzschild, 1916), soon after Einstein introduced General Relativity (GR). The first solution describes the geometry of the space-time exterior to a prefect fluid sphere in hydrostatic equilibrium. While the other, known as

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interior Schwarzschild solution, corresponds to the interior geometry of a fluid sphere of constant (homogeneous) energy-density, E. The importance of these two solutions in GR is well known. The exterior solution at a given point depends only upon the total mass of the gravitating body and the radial distance as measured from the centre of the spherical symmetry, and not upon the 'type' of the density distribution considered inside the mass. However, we will focus on this point of crucial importance later on in the present paper. On the other hand, the interior Schwarzschild solution provides two very important features towards obtaining configurations in hydrostatic equilibrium, compatible with GR, namely - (i) It gives an absolute upper limit on compaction parameter, $u(\equiv M/a$, mass to size ratio of the entire configuration in geometrized units) $\leq (4/9)$ for any static and spherical solution (corresponding to arbitrary density profile as long as to spheres in which the density does not increase outwards) in hydrostatic equilibrium (Buchdahl, 1959; Weinberg, 1972), and (ii) For an assigned value of compaction parameter, u and radius a (or mass M), the minimum central pressure, P_0, corresponds to the homogeneous density solution (see, e.g., Weinberg, 1972)). Regarding these conditions, it should be noted that the condition (i) tells us that the values higher than the limiting (maximum) value of $u(= 4/9)$ can not be attained by any static solution. But, what kinds of density variations are possible for a mass to be in the state of hydrostatic equilibrium?, the answer to this important question could be provided by an appropriate analysis of the condition (ii), and the necessary conditions put forward by exterior Schwarzschild solution.

Despite the non linear differential equations, various exact solutions for static and spherically symmetric metric are available in the literature (Kramer *et al.*, 1980). Tolman (1939) obtained five different types of exact solutions for static cases, namely - type III (which corresponds to the constant density solution obtained earlier by Schwarzschild (1916)), type IV, type V, type VI, and type VII. The solution independently obtained by Adler (1974), Adams and Cohen (1974), and Kuchowicz (1975). Buchdahl's solution (1967) for vanishing surface density (the "gaseous" model). The solution obtained by Vaidya and Tikekar (1982), which is also obtained independently by Durgapal and Bannerji (1983). The class of exact solutions discussed by Durgapal (1982), and also Durgapal and Fuloria (1985) solution. Knutsen (1988a,b, 1989) examined various physical properties of the solutions mentioned in references (Adler, 1974; Adams and Cohen, 1974; Kuchowicz 1975; Vaidya and Tikekar, 1982; Durgapal and Bannerji, 1983; Durgapal and Fuloria, 1985) in great detail, and found that these solutions correspond to nice physical properties and also remain stable against small radial pulsations upto certain values of u. Tolman's V and VI solutions are not considered physically viable, as they correspond to singular solutions [infinite values of central density (that is, the metric coefficient, $e^\lambda \neq 1$ at $r = 0$) and pressure for all permissible values of u]. Except Tolman's V and VI solutions, all other

solutions mentioned above are known as regular solutions [finite positive density at the origin (that is, the metric coefficient, $e^\lambda = 1$ at $r = 0$) which decreases monotonically outwards], which can be further divided into two categories: (i) regular solutions corresponding to a vanishing density at the surface together with pressure (like, Tolman's VII solution (Mehra, 1966); Durgapal and Rawat, 1980; Negi and Durgapal 1996, 1999), and Buchdahl's "gaseous" solution[1] (Buchdahl, 1967)), and (ii) regular solutions correspond to a non-vanishing density at the surface (like, Tolman's III and IV solutions (Tolman, 1939), and the solutions discussed in Adler, (1974), Adams and Cohen, (1974); Kuchowicz, (1975), Vaidya and Tikekar, (1982), Durgapal and Bannerji, (1983), Durgapal, 1982; Durgapal and Fuloria, (1985) respectively).

The stability analysis of Tolman's VII solution with vanishing surface density has been undertaken in detail by Negi and Durgapal (1996, 1999a) and they have shown that this solution also corresponds to stable Ultra-Compact Objects (UCOs) which are entities of physical interest. This solution also shows nice physical properties, such as, pressure and energy-density are positive and finite everywhere, their respective gradients are negative, the ratio of pressure to density and their respective gradients decrease outwards etc. The other solution which falls in this category and shows nice physical properties is the Buchdahl's solution (1967), however, Knutsen (1988) has shown that this solution turned out to be unstable under small radial pulsations for $0 < u \leq (1/6)$.

All these solutions (with finite, as well as vanishing surface density) discussed above, in fact, fulfill the condition (i), that is, the equilibrium configurations pertaining to these solutions always correspond to a value of compaction parameter, u, which is always less than the Schwarzschild limit, i. e., $u \leq (4/9)$, but, this condition alone does not provide a necessary condition for hydrostatic equilibrium. Nobody has discussed until now, whether these solutions also fulfill the condition (ii)? which is necessary to satisfy by any static and spherical configuration in the state of hydrostatic equilibrium!

Recently, by using the condition (ii), we have connected the compaction parameter, u, of any static and spherical configuration with the corresponding ratio of central pressure to central energy-density $\sigma [\equiv (P_0/E_0)]$ and worked out an important criterion which concludes that for a given value of σ, the maximum value of compaction parameter, $u(\equiv u_h)$, should always correspond to the homogeneous density sphere (Negi and Durgapal, 2001). An examination of this criterion on some well known exact solutions and equations of state (EOSs) indicated that this criterion, in fact, is fulfilled only by those regular configurations which correspond to a vanishing density at the surface together with pressure (Negi and Durgapal, 2001). In continuation of this study, in the present paper, we have examined

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[1] This solution satisfies the condition of regularity for the values of $u \leq 0.20$ which is discussed under Section 5(e) of the present study.

various exact solutions available in the literature in detail. It is seen that Tolman's VII solution with vanishing surface density (Mehra, 1966; Negi and Durgapal, 1996, 1999a), Buchdahl's "gaseous" solution (Buchdahl, 1967), and Tolman's V and VI solutions [singular solutions with non-vanishing surface density] pertain to a value of u which always turns out to be less than the value, u_h, of the homogeneous density sphere for all assigned values of σ. On the other hand, the regular solutions having a finite non-zero surface density (that is, the pressure vanishes at the finite surface density) do not show consistency with the structure of the general relativity, as they correspond to a value of u which turns out to be greater than u_h for all assigned values of σ, and thus violate the criterion obtained in Negi and Durgapal, (2001) (see also Negi, 2004).

This result has motivated us to investigate and disclose the reason (s) behind non-fulfillment of the said criterion by various regular analytic solutions and EOSs corresponding to a non-vanishing finite density at the surface of the configuration. We have been able to pin point (which is discussed under Section 3 of the present study) the main reason, namely, the 'actual' total mass 'M' which appears in the exterior Schwarzschild solution, in fact, can not be attained by the configurations corresponding to a regular density variation with non-vanishing surface density.

2. FIELD EQUATIONS AND EXACT SOLUTIONS

The metric inside a static and spherically symmetric mass distribution corresponds to

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{1}$$

where ν and λ are functions of r alone. The resulting field equations for the metric governed by Eq. (1) yield in the following form

$$8\pi T_0^0 = 8\pi E = e^{-\lambda}[(\lambda'/r) - (1/r^2)] + 1/r^2, \tag{2}$$

$$-8\pi T_1^1 = 8\pi P = e^{-\lambda}[(\nu'/r) - (1/r^2)] - 1/r^2, \tag{3}$$

$$-8\pi T_2^2 = -8\pi T_3^3 = 8\pi P = e^{-\lambda}[(\nu''/2)$$
$$+ (\nu'^2/4) - (\nu'\lambda'/4) + (\nu' - \lambda')/2r]. \tag{4}$$

where the primes represent differentiation with respect to r, the speed of light c and the Universal gravitation constant G are chosen as unity (that is, we are using the geometrized units). P and E represent, respectively, the pressure and energy-density inside the perfect fluid sphere related with the non-vanishing components of the energy-momentum tensor, $T_i^j = 0$, 1, 2, and 3 respectively.

Equations (2)–(4) represent second-order, coupled differential equations which can be reduced to the first-order, coupled differential equations, by eliminating ν'' from Eq. (3) with the help of Eqs. (2) and (4) in the well known

form (namely, TOV equations (Tolman, 1939; Oppenheimer and Volkoff 1939), governing hydrostatic equilibrium in general relativity)

$$P' = -(P + E)[4\pi P r^3 + m(r)]/r(r - 2m(r)), \tag{5}$$

$$\nu' = -2P'/(P + E), \tag{6}$$

and

$$m'(r) = 4\pi E r^2, \tag{7}$$

where prime denotes differentiation with respect to r, and $m(r)$ is defined as the mass-energy contained within the radius 'r', that is

$$m(r) = \int_0^r 4\pi E r^2 dr. \tag{8}$$

The equation connecting metric parameter λ with $m(r)$ is given by

$$e^{-\lambda} = 1 - [2m(r)/r] = 1 - (8\pi/r) \int_0^r E r^2 dr. \tag{9}$$

The three field equations (or TOV equations) mentioned above, involve four variables, namely, P, E, ν, and λ. Thus, in order to obtain a solution of these equations, one more equation is needed [which may be assumed as a relation between P, and E (EOS), or can be regarded as an algebraic relation connecting one of the four variables with the radial coordinate r (or an algebraic relation between the parameters)]. For obtaining an exact solution, the later approach is employed.

Notice that Eq. (9) yields the metric coefficient e^λ for the assumed energy-density, E, as a function of radial distance 'r'. Once the metric coefficient e^λ or mass $m(r)$ is defined for assumed energy-density by using Eqs. (9) or (8), the pressure, P, and the metric coefficient, e^ν, can be obtained by solving Eqs. (5) and (6) respectively which yield two constants of integration. These constants should be obtained from the following boundary conditions, in order to have a proper solution of the field equations:

3. BOUNDARY CONDITIONS: THE VALID AND INVALID ASSUMPTIONS FOR MASS DISTRIBUTION

(B1) In order to maintain hydrostatic equilibrium throughout the configuration, the pressure must vanish at the surface of the configuration, that is

$$P = P(r = a) = P_a = 0, \tag{10}$$

where 'a' is the radius of the configuration.

(B2) The consequence of Eq. (10) ensures the continuity of the metric parameter e^ν, belonging to the interior solution with the corresponding expression

for well known exterior Schwarzschild solution at the surface of the fluid con-
figuration, that is: $e^{\nu(r=a)} = 1 - (2M/a)$ [where 'M' $= m(a)$ is the total mass of
the configuration]. However, the exterior Schwarzschild solution guarantees that:
$e^{\nu(r=a)} = e^{-\lambda(r=a)}$, which means that the matching of the metric parameter e^{λ} is
also ensured at the surface of the configuration together with e^{ν}, that is

$$e^{\nu(a)} = e^{-\lambda(a)} = 1 - (2M/a) = (1 - 2u), \tag{11}$$

irrespective of the condition that the surface density, $E_a = E(r = a)$, is vanishing
with pressure or not, that is

$$E_a = 0. \tag{12}$$

together with Eq. (10), or

$$E_a \neq 0, \tag{13}$$

where 'u' is the 'compaction parameter' of the configuration defined earlier, and
'M' is defined as (Eq. (8))

$$M = m(a) = \int_0^a 4\pi \, E r^2 \, dr. \tag{14}$$

Thus, the analytic solution for the fluid sphere can be explored in terms of the
only free parameter 'u' by normalizing the metric coefficient e^{λ} [yielding from Eq.
(11)] at the surface of the configuration [that is, $e^{\lambda} = (1 - 2u)^{-1}$ at $r = a$] after
obtaining the integration constants by using Eqs. (10) [that is, $P = 0$ at $r = a$]
and (11) [$e^{\nu} = (1 - 2u)$ at $r = a$] respectively.

However, at this place we recall the well known property of the exterior
Schwarzschild solution (which follows directly from the definition of the mass, M,
appears in this solution), namely - at a given point outside the spherical distribution
of mass, M, it depends only upon M, and not upon the 'type' of the density
variation considered inside the radius, a, of this sphere. It follows, therefore, that
the dependence of mass, M, upon the 'type' of the density distribution plays an
important role in order to fulfill the requirement set up by exterior Schwarzschild
solution. The relation, $M = au$, immediately tells us that for an assigned value
of the compaction parameter, u, the mass, M, depends only upon the radius,
a, of the configuration which may either depend upon the surface density, or
upon the central density, or upon both of them, depending upon the 'type' of the
density variation considered inside the mass generating sphere. We argue that this
dependence should occur in such a manner that the definition of mass, M, is not
violated. We infer this definition as the 'type independence' property of the mass,
M, which may be defined in this manner: "The mass, M, which appears in the
exterior Schwarzschild solution, should either depend upon the surface density, or
upon the central density, and in any case, not upon both of them so that from an

exterior observer's point of view, the 'type' of the density variation assigned for the mass should remain unidentified".

We may explain the 'type independence' property of mass, M, mentioned above in the following manner: The mass, M, is called the coordinate mass, that is, the mass as measured by some external observer, and from this observer's point of view, if we are 'measuring' a sphere of mass, M, we can not know, by any means, the way in which the matter is distributed from the centre to the surface of this sphere, that is, if we are measuring, M, with the help of non-vanishing surface density [obviously, by calculating the (coordinate) radius, a, from the expression connecting the surface density and the compaction parameter, and by using the relation, $M = au$], we can not measure it, by any means, from the knowledge of the central density (because, if we can not know, by any means, the way in which the matter is distributed from the centre to the surface of the configuration, then how can we know about the central density?), and this is possible only when there exist no relation connecting the mass, M, and the central density, that is, the mass, M, should be independent of the central density, meaning thereby that *the surface density should be independent of the central density for configurations corresponding to a non-vanishing surface density.* However, if we are measuring the mass, M, by using the expression for central density (in the similar manner as in the previous case, by calculating the radius, a, from the expression of central density, and using the relation, $M = au$), we can not calculate it, by any means, from the knowledge of the surface density (in view of the 'type independence' property of the mass, M), and this is possible only when there exist no relation connecting the mass, M, and the surface density, meaning thereby that the central density should be independent of the surface density.

From the above explanation of 'type independence' property of mass 'M', it is evident that *the 'actual' total mass 'M' which appears in the exterior Schwarzschild solution should either depend upon the surface density, or depend upon the central density of the configuration, and in any case, not upon both of them.* However, the dependence of mass 'M' upon both of the densities (surface, as well as central) is a common feature observed among all regular solutions having a non-vanishing density at the surface of the configuration [see, for example, Eqs. (21), (25), (29), and (33) respectively, belonging to the solutions of this category which are discussed under subsections (a)–(d) of Section 5 of the present study]. Thus, it is evident that the surface density of such solutions is *dependent* upon the central density and vice-versa, that is, the total mass, M, depends upon *both* of the densities, meaning thereby that *the 'type' of the density distribution considered inside the sphere of mass, M, is known to an external observer* which is the violation of the definition of mass, M (defined as the 'type independence' property of mass 'M' above), such structures, therefore, do not correspond to the 'actual' total mass 'M' required by the exterior Schwarzschild solution to ensure the condition of hydrostatic equilibrium. This also explains the reason behind non-fulfillment

of the 'compatibility criterion' by them which is discussed under Section 5 of the present study. However, it is interesting to note here that there could exist only one solution in this regard for which the mass 'M' depends upon both, but the same, value of surface and centre density, and for regular density distribution the structure would be governed by the homogeneous (constant) density throughout the configuration (that is, the homogeneous density solution).

Note that the requirement, 'type independence' of the mass would be obviously fulfilled by the regular structures corresponding to a vanishing density at the surface together with pressure, because the mass 'M' will depend only upon the central density (surface density is always zero for these structures) [see, for example, Eqs. (37) and (41), discussed under sub-section (e) and (f) for Buchdahl's "gaseous" model and Tolman's VII solution having a vanishing density at the surface, respectively].

Furthermore, the demand of 'type independence' of mass, M, is also satisfied by the singular solutions having a non-vanishing density at the surface, because such structures correspond to an infinite value of central density, and consequently, the mass 'M' will depend only upon the surface density [see, for example, Eqs. (46) and (50), discussed under sub-section (g) and (h) for Tolman's V and VI solutions, respectively]. Both types of these structures are also found to be consistent with the 'compatibility criterion' as discussed under Section 5 of the present study.

The discussion regarding various types of density distributions considered above is true for any single analytic solution or equation of state comprises the whole configuration. At this place, we are not intended to claim that the construction of a regular structure with non-vanishing surface density is impossible. It is quite possible, provided we consider a two-density structure in such a manner that the mass 'M' of the configuration turns out to be independent of the central density so that the property 'type independence' of the mass 'M' is satisfied. Examples of such two-density models are also available in the literature (see, e.g., (Negi et al. 1989, 1990)), but in the different context. However, it should be noted here that the fulfillment of 'type independence' condition by the mass 'M' for any two-density model will represent only a necessary condition for hydrostatic equilibrium, unless the 'compatibility criterion' (Negi and Durgapal, 2001) is satisfied by them, which also assure a sufficient and necessary condition for any structure in hydrostatic equilibrium (this issue is addressed in the next section of the present study).

The above discussion can be summarized in other words as: although the exterior Schwarzschild solution itself does not depend upon the type of the density distribution or EOS considered inside a fluid sphere in the state of hydrostatic equilibrium, however, it puts the important condition that only two types of the density variations are possible inside the configuration in order to fulfill the condition of hydrostatic equilibrium: (1) the surface density of the configuration should be independent of the central density, and (2) the central density of the configuration should be independent of the surface density. Obviously, the condition (1) will be

satisfied by the configurations pertaining to an infinite value of the central density (that is, the singular solutions), and/or by the two-density (or multiple-density) distributions corresponding to a surface density which turns out to be independent of the central density (because, the regular configurations governed by a single exact solution or EOS pertaining to this category are not possible). Whereas, the condition (2) will be fulfilled by the configurations corresponding to a surface density which vanishes together with pressure [the configurations in this category will include: the density variation governed by a single exact solution or EOS, as well as the two-density (see, e.g. (Negi, 2005) for a two-density distribution of this category) (or multiple-density) distributions]. However, the point to be emphasized here is that a two-density distribution in any of the two categories mentioned here will fulfill only a *necessary* condition for hydrostatic equilibrium unless the 'compatibility criterion' (Negi and Durgapal, 2001) is satisfied by them which also assure a necessary and sufficient condition for any structure in the state of hydrostatic equilibrium as mentioned above.

4. CRITERION FOR STATIC SPHERICAL CONFIGURATIONS TO BE CONSISTENT WITH THE STRUCTURE OF GENERAL RELATIVITY

The criterion obtained in Negi and Durgapal (2001) can be summarized in the following manner: For an assigned value of the ratio of central pressure to central energy-density $\sigma [\equiv (P_0/E_0)]$, the compaction parameter of homogeneous density distribution, $u(\equiv u_h)$ should always be larger than or equal to the compaction parameter $u(\equiv u_v)$ of any static and spherical solution, compatible with the structure of General Relativity. That is

$$u_h \geq u^v \text{ (for an assigned value of } \sigma). \tag{15}$$

In the light of Eq. (15), let us assign the same value, M, for the total mass corresponding to various static configurations in hydrostatic equilibrium. If we denote the density of the homogeneous sphere by E_h, we can write

$$E_h = 3M/(4\pi a_h{}^3) \tag{16}$$

where a_h, denotes the radius of the homogeneous density sphere. If a_v represents the radius of any other regular sphere for the same mass, M, the average density, E_v, of this configuration would correspond to

$$E_v = 3M/(4\pi a_v{}^3). \tag{17}$$

Equation (15) indicates that $a_v \geq a_h$. By the use of Eqs. (16) and (17) we find that

$$E_v \leq E_h. \tag{18}$$

That is, for an assign value of σ, the average energy-density of any static spherical configuration, E_v, should always be less than or equal to the density, E_h, of the homogeneous density sphere for the same mass, M.

Although, the regular configurations with finite non-vanishing surface densities, represented by a single density variation can not exist, because for such configurations the necessary condition set up by exterior Schwarzschild solution can not be satisfied, however, we can construct regular configurations composed of core-envelope models corresponding to a finite central with vanishing and non-vanishing surface densities, such that the necessary conditions imposed by the Schwarzschild's exterior solution at the surface of the configuration are appropriately satisfied. However, it should be noted that the necessary conditions satisfied by such core-envelope models at the surface may not always turn out to be sufficient for describing the state of hydrostatic equilibrium [because for an assigned value of σ, the average density of such configurations may not always turn out to be less than or equal to the density of the homogeneous density sphere for the same mass, as indicated by Eqs. (16) and (17) respectively (it would depend upon the types of the density variations considered for the core and envelope regions and the the matching conditions at the core-envelope boundary)]. Thus, it follows that the criterion obtained in (Negi and Durgapal, 2001) is able to provide a *necessary* and *sufficient* condition for any regular configuration to be consistent with the state of hydrostatic equilibrium.

The future study of such core-envelope models (see, for example, the models described in Negi et al. (1989, 1990) and Negi and Durgapal, (1999b), based upon the criterion obtained in Negi and Durgapal, (2001) could be interesting regarding two density structures of neutron stars and other stellar objects compatible with the structure of GR.

5. EXAMINATION OF THE COMPATIBILITY CRITERION FOR VARIOUS WELL KNOWN EXACT SOLUTIONS AVAILABLE IN THE LITERATURE

We have considered the following exact solutions expressed in units of compaction parameter $u[\equiv (M/a)$, mass to size ratio in geometrized units], and radial coordinate measured in units of configuration size, $y[\equiv (r/a)]$, for convenience. The other parameters which will appear in these solutions, are defined at the relevant places. In these equations, P and E represent, respectively, the pressure and energy-density inside the configuration. The surface density is denoted by E_a, and the central pressure and central energy-density are denoted by P_0, and E_0, respectively.

The regular exact solutions which pertain to a non-vanishing value of the surface density are given under the sub-sections (a)–(d), while those correspond to a vanishing value of the surface density are described under the sub-sections

(e) and (f) respectively. Sub-sections (g) and (h) represent the singular solutions having non-vanishing values of the surface densities.

(a) Tolman's IV solution

$$8\pi P a^2 = \frac{3u^2(1 - y^2)}{(1 - 3u + 2uy^2)}, \tag{19}$$

$$8\pi E a^2 = \frac{u(4 - 9u + 3uy^2)}{1 - 3u + 2uy^2} + \frac{2u(1 - 3u)(1 - uy^2)}{(1 - 3u + 2uy^2)^2}. \tag{20}$$

By the use of Eq. (20), we can obtain the relation connecting central and surface densities in the following form

$$(E_a/E_0) = \frac{2(1 - 2u)(1 - 3u)}{(1 - u)(2 - 3u)}. \tag{21}$$

Equation (21) shows that the surface density is *dependent* upon the central density, and vice-versa.

By using Eqs. (19) and (20), we obtain

$$(P_0/E_0) = u/(2 - 3u). \tag{22}$$

This solution finds application [it can be seen from Eq. (19)] for the values of $u \leq (1/3)$.

(b) Adler (1974), Adams and Cohen (1974), and Kuchowicz (1975) solution:

$$8\pi P a^2 = \frac{2}{(1 + Z)} \left[2x - \frac{u(1 + 5z)(1 + 3x)^{2/3}}{(1 + 3z)^{2/3}} \right], \tag{23}$$

$$8\pi E a^2 = \frac{2u(3 + 5z)(1 + 3x)^{2/3}}{(1 + 5z)^{5/3}}, \tag{24}$$

where $z = xy^2$, and $u = 2x/(1 + 5x)$.

Equation (24) gives the relation, connecting central and surface densities of the configuration in the following form

$$(E_a/E_0) = \frac{(3 + 5x)}{3(1 + 5x)^{5/3}}. \tag{25}$$

Thus, the surface density *depends* upon the central density and vice-versa.

Equations (23) and (24) give

$$(P_0/E_0) = (1/3) \left[\frac{(1 + 5x)}{(1 + 3x)^{2/3}} - 1 \right]. \tag{26}$$

It is seen from Eq. (23) that this solution finds application for values of $u \leq (2/5)$.

(c) Vaidya and Tikekar (1982), and Durgapal and Bannerji (1983) solution

$$8\pi P = (9C/2)\frac{(1+x)^{1/2}(1-x) - B(1+2x)(2-x)^{1/2}}{(1+x)[(1+x)^{3/2} + B(2-x)^{1/2}(5+2x)]}, \quad (27)$$

$$8\pi E = (3C/2)\frac{(3+x)}{(1+x)^2}. \quad (28)$$

where the variable x, and the constants B and C are given by

$x = Cr^2; X = Ca^2 = 4u/(3-4u)$; and

$$B = \frac{(1+X)^{1/2}(1-X)}{(1+2X)(2-X)^{1/2}}.$$

By the use of Eq. (28), we find that the surface and central densities are connected by the following relation

$$(E_a/E_0) = \frac{(3+X)}{3(1+X)^2}. \quad (29)$$

It is evident from Eq. (29) that the surface density is *dependent* upon the central density, and vice-versa.

By the use of Eqs. (27) and (28), we obtain

$$(P_0/E_0) = \frac{(1 - B\sqrt{2})}{(1 + 5B\sqrt{2})}. \quad (30)$$

This solution finds application for the values of $u \leq 0.4214$, as shown by Eq. (27).

(d) Durgapal and Fuloria (1985) solution

$$\frac{8\pi P}{C} = \frac{16}{7z(1+x)^2}[(1+x)(2-7x-x^2)$$

$$- A[(18 + 25x + 3x^2)(7 - 10x - x^2)^{1/2}$$

$$- 4(1+x)(2-7x-x^2)W(x)]], \quad (31)$$

$$\frac{8\pi E}{C} = \frac{8(9+2x+x^2)}{7(1+x)^3}. \quad (32)$$

where C is a constant and $x = Cr^2$.

The variables z and $W(x)$ are given by

$$z = (1+x)^2 + A[(7+3x)(7-10x-x^2)^{1/2} + 4W(x)(1+x)^2],$$

$$W(x) = \ln\left[\frac{(3-x) + (7-10x-x^2)^{1/2}}{(1+x)}\right],$$

where the arbitrary constant A is given by

$$A = \frac{(1 + X)Y}{[B(7 - 10X - X^2)^{1/2} - 4Y(1 + X)W(X)]}$$

and $X = Ca^2$; $Y = (2 - 7X - X^2)$; $B = (18 + 25X + 3X^2)$

Equation (32) gives the relation, connecting central and surface densities as

$$(E_a/E_0) = \frac{(9 + 2X + X^2)}{9(1 + X)^3}. \tag{33}$$

Equation (33) indicates that the surface density is *dependent* upon the central density, and vice-versa.

By the use of Eq. (31) and (32), we get

$$(P_0/E_0) = \frac{2}{9z(0)}[2 - A[18\sqrt{7} - 8W(0)]], \tag{34}$$

where $z(0)$ and $W(0)$ are given by

$$z(0) = 1 + A[7\sqrt{7} + 4W(0)],$$

$$W(0) = \text{In } (3 + \sqrt{7}),$$

and

$$u = \frac{8X(3 + X)}{14(1 + X)^2}.$$

As seen from Eq. (31), this solution is applicable for the values of $u \leq 0.4265$.

(e) Buchdahl's "gaseous" solution (Buchdahl, 1967)

$$8\pi Pa^2 = \frac{\pi^2(1 - u)^2}{(1 - 2u)} \frac{m^2}{(n + m)^2}, \tag{35}$$

$$8\pi Ea^2 = \frac{\pi^2(1 - u)^2}{(1 - 2u)} \frac{m(2n - 3m)}{(n + m)^2}, \tag{36}$$

where $m = 2u[\sin(z)/z]$; $n = 2(1 - u)$,
and

$$z = \frac{\pi y}{[1 + u(\sin(z)/z)(1 - u)^{-1}]}; \quad 0 \leq z \leq \pi$$

Equation (36) shows that the surface density vanishes together with pressure, thus, the central density will become *independent* of the surface density, given by the equation

$$8\pi E_0 a^2 = \frac{\pi^2 u(2 - 5u)(1 - u)^2}{(1 - 2u)}. \tag{37}$$

By using Eq. (35) and (36), we obtain

$$(P_0/E_0) = u/(2 - 5u). \tag{38}$$

It is evident from Eq. (35) that this solution is applicable for the values of $u \leq (2/5)$. However, the density, E, must not increase outward, that is, the condition $E' \leq 0$ requires that $u \leq 0.20$ (Eq. (36)). Thus, the results for the values $0.2 < u \leq (2/5)$ as shown in Table 2 are only academic.

(f) Tolman's VII solution with vanishing surface density

$$8\pi P a^2 = La^2 \frac{C_2 \cos(w/2) - C_1 \sin(w/2)}{C_1 \cos(w/2) + C_2 \sin(w/2)} - Na^2, \tag{39}$$

$$8\pi E a^2 = 8\pi E_0 a^2 (1 - x), \tag{40}$$

where E_0 is the central energy-density given by

$$8\pi E_0 a^2 = 15u. \tag{41}$$

and

$x = (r/a)^2 = y^2,$

$C_1 = A \cos(w_a/2) - B \sin(w_a/2); C_2 = A \sin(w_a/2) + B \cos(w_a/2),$

$La^2 = 2(3u)^{1/2}[1 - ux(5 - 3x)]^{1/2}; Na^2 = u(5 - 3x),$

$w = \ln [x - (5/6) + ([1 - ux(5 - 3x)]/3u)^{1/2}],$

$w_a = $ (the value of w at $y = 1$) $= \ln [(1/6) + [(1 - 2u)/3u]^{1/2}],$

$A = (1 - 2u)^{1/2}; \quad B = (u/3)^{1/2}.$

By using Eqs. (39) and (40), we get

$$(P_0/E_0) = (1/3)[(2/5)(3/u)^{1/2} \frac{C_2 \cos(w/2) - C_1 \sin(w/2)}{C_1 \cos(w/2) + C_2 \sin(w/2)} - 1], \tag{42}$$

where w_0 is given by

$$w_0 = \ln [(1/3u)^{1/2} - (5/6)].$$

It follows from Eq. (40) that the surface density is always zero, hence the central density is always *independent* of the surface density.

Equation (39) indicates that his solution is applicable for the values of $u \leq 0.3861$.

(g) Tolman's V solution

$$8\pi P a^2 = \frac{n^2(1 - y^q)}{(2n + 1 - n^2)y^2}, \tag{43}$$

and

$$8\pi E a^2 = n \frac{[(2-n) + [n(3-n)/(1+n)]y^q]}{(2n+1-n^2)y^2}, \tag{44}$$

where q is given by

$$q = 2(2n+1-n^2)/(1+n),$$

and n is defined as

$$n = u/(1-2u).$$

Equation (44) shows that the central density is always infinite (for $n < 2$) together with central pressure (Eq. (43)), however, their ratio (P/E) is finite at all points inside the configuration, and at the centre, yields in the following form

$$(P_0/E_0) = n/(2-n) = u/(2-5u). \tag{45}$$

The consequence of the infinite central density is that the surface density will become *independent* of the central density, given by the equation

$$8\pi E_a a^2 = 2n/(n+1) = 2u/(1-u). \tag{46}$$

It is evident from Eq. (45) that this solution is applicable for a value of $n \leq 2$ [i. e., for a value of $u \leq (2/5)$].

(h) Tolman's VI solution

$$8\pi P a^2 = \frac{(1-n^2)^2(1-y^{2n})}{y^2(2-n^2)[(1+n)^2 - y^{2n}(1-n)^2]}, \tag{47}$$

$$8\pi E a^2 = \frac{(1-n^2)}{y^2(2-n^2)}. \tag{48}$$

Equation (47) and (48) indicate that the central pressure and central density are always infinite, however, their ratio (P/E) is finite at all points inside the structure, and at the centre, reduces into the following form

$$(P_0/E_0) = (1-n^2)/(1+n)^2 = (1-n)/(1+n), \tag{49}$$

and the surface density (obviously, *independent* of the central density) would be given by the equation

$$8\pi E_a a^2 = \frac{(1-n)^2}{(2-n^2)} = 2u. \tag{50}$$

where n is defined as

$$n^2 = (1-4u)/(1-2u).$$

Equation (49) indicates that this solution is applicable for a value of $u \leq (1/4)$.

Let us denote the compaction parameter of the homogeneous density configuration by u_h, and for the exact solutions corresponding to the sub-sections (a) – (d) by u_{IV}, u_{AAK}, u_{DBN}, and u_{DFN} respectively. The compaction parameters of the exact solutions described under sub-section (e) and (f) are denoted by u_{BDL} and u_{VII} respectively, and those discussed under sub-sections (g) and (h) are denoted by u_{V} and u_{VI} respectively.

Solving these analytic solutions for various assigned values of the ratio of central pressure to central energy-density $\sigma [\equiv (P_0/E_0)]$, we obtain the corresponding values of the compaction parameters as shown in Tables I and II respectively. It is seen that for each and every assigned value of σ, the values represented by u_{IV}, u_{AAK}, u_{DBN}, and u_{DFN} respectively (Table I), turn out to be higher than u_h (that is, u_{IV}, u_{AAK}, u_{DBN}, and $u_{\text{DFN}} \geq u_h$), while those represented by u_{BDL}, u_{VII}, u_{V}, and u_{VI} respectively (Table II) correspond to a value which always remains less than u_h (that is, u_{BDL}, u_{VII}, u_{V}, and $u_{\text{VI}} \leq u$). Thus, the configurations defined by u_{IV}, u_{AAK}, u_{DFN}, and u_{DBN} respectively, do not show compatibility with the structure of general relativity, while those defined by u_{BDL}, u_{VII}, u_{V}, and u_{VI} respectively, show compatibility with the structure of general relativity. However, this type of characteristics [that is, the value of compaction parameter larger than the value of u_h for some or all assigned values of σ] can be seen for any regular exact solution having a finite non-vanishing surface density [because such exact solutions (having finite central densities) with non-vanishing surface densities can not possess the actual mass, M, required to fulfill the boundary conditions at the surface]. On the other hand, the value of compaction parameter for a regular solution with vanishing surface density, and a singular solution with non-vanishing surface density will always remain less than the value of u_h for all assigned values of σ [because such solutions naturally fulfill the definition of the actual mass, M, required for the hydrostatic equilibrium].

Therefore, it is evident that the findings based upon the 'compatibility criterion' carried out in this section are fully consistent with the definition of the mass, M (defined as the 'type independence' property under Section 3 of the present study).

6. RESULTS AND CONCLUSIONS

We have investigated the criterion obtained in the reference (Negi and Durgapal, 2001) which states: for an assigned value of the ratio of central pressure to central energy-density, $\sigma (\equiv P_0/E_0)$, the compaction parameter, $u(\equiv M/a)$, of any static and spherically symmetric solution should always be less than or equal to the compaction parameter, u_h, of the homogeneous density distribution. We conclude that this criterion is fully consistent with the reasoning discussed under

Section 3 which states that in order to fulfill the requirement set up by exterior Schwarzschild solution (that is, to ensure the condition of hydrostatic equilibrium), the total mass, M, of the configuration should depend either upon the surface density (that is, independent of the central density), or upon the central density (that is, independent of the surface density), and in any case, not upon both of them.

An examination, based upon this criterion, show that among various exact solutions of the field equations available in the literature, the regular solutions corresponding to a vanishing surface density together with pressure, namely - (i) Tolman's VII solution with vanishing surface density, and (ii) Buchdahl's

Table I. Various values (round off at the fourth decimal place) of the compaction parameter $u(\equiv M/a)$ as obtained for different assigned values of the ratio of the centre pressure to centre energy-density, $\sigma[\equiv (P_0/E_0)]$, corresponding to the regular exact solutions with finite non-vanishing surface densities, namely - Tolman's IV (Tolman, 1939) solution [indicated by u_{IV}], Adler (1974), Adams and Cohen (1974), and Khchowicz's (1975) solution [indicated by u_{AAK}, Vaidya and Tikekar (1982), and Durgapal and Bannerji (1983) solution [indicated by u_{DBN}], and Durgapal and Fuloria (1985) solution [indicated by u_{DFN}] respectively

$\sigma(\equiv P_0/E_0)$	u_h	u_{IV}	u_{AAK}	u_{DBN}	u_{DFN}
0.1252	0.1654	0.1820	0.1745	0.1743	0.1718
0.1859	0.2102	0.2387	0.2242	0.2221	0.2187
0.2202	0.2301	0.2652	0.2463	0.2429	0.2392
0.2800	0.2580	0.3043	0.2771	0.2714	0.2676
0.3150	0.2714	0.3239	0.2917	0.2847	0.2809
(1/3)	0.2778	(1/3)	0.2984	0.2909	0.2872
0.3774	0.2914		0.3127	0.3038	0.3003
0.4350	0.3062		0.3277	0.3176	0.3145
0.4889	0.3178		0.3390	0.3281	0.3253
0.5499	0.3289		0.3493	0.3378	0.3354
0.6338	0.3415		0.3600	0.3485	0.3465
0.6830	0.3476		0.3650	0.3535	0.3519
0.7044	0.3501		0.3669	0.3555	0.3541
0.7085	0.3506		0.3673	0.3559	0.3545
0.7571	0.3558		0.3711	0.3601	0.3589
0.8000	0.3599		0.3740	0.3633	0.3624
0.8360	0.3630		0.3762	0.3658	0.3650

The compaction parameter corresponding to homogeneous density distribution (Schwarzschild's interior solution) is indicated by u_h for the same value of σ. It is seen that for each and every assigned value of σ, u_{IV}, u_{AAK}, u_{DBN}, and $u_{DFN} > u_h$ which is the evidence that the regular solutions corresponding to a finite non-vanishing surface density (indicated by u_{IV}, u_{AAK}, u_{DBN}, and u_{DFN} respectively) are not compatible with the structure of general relativity

Table II. Various values (round off at the fourth decimal place) of the compaction parameter $u(\equiv M/a)$ as obtained for different assigned values of the ratio of the centre pressure to centre energy-density, $\sigma\,[\equiv (P_0/E_0)]$, corresponding to the regular exact solutions with vanishing surface densities, namely - Buchdahl's (1967) "gaseous" solution and Tolman's VII solution (Tolman, 1939; Mehra, 1966; Durgapal and Rawat, 1980; Negi and Durgapal, 1996; Negi and Durgapal, 1999) (indicated by u_{BDL} and u_{VII} respectively), and singular solutions with non-vanishing surface densities, namely - Tolman's V and VI solutions (indicated by u_{V} and u_{VI} respectively)

$\sigma(\equiv P_0/E_0)$	u_h	u_{BDL}	u_{VII}	u_{V}	u_{VI}
0.1252	0.1654	0.1540	0.1588	0.1540	0.1417
0.1859	0.2102	0.1927	0.1992	0.1927	0.1730
0.2202	0.2301	0.2096	0.2166	0.2096	0.1858
0.2800	0.2580	0.2333	0.2407	0.2333	0.2031
0.3150	0.2714	0.2446	0.2521	0.2447	0.2108
(1/3)	0.2778	0.2500	0.2574	0.2500	0.2143
0.3774	0.2914	0.2614	0.2687	0.2614	0.2216
0.4350	0.3062	0.2740	0.2809	0.2740	0.2290
0.4889	0.3178	0.2839	0.2904	0.2839	0.2344
0.5499	0.3289	0.2934	0.2993	0.2934	0.2390
0.6338	0.3415	0.3040	0.3092	0.3041	0.2436
0.6830	0.3476	0.3094	0.3140	0.3094	0.2455
0.7044	0.3501	0.3115	0.3160	0.3115	0.2462
0.7085	0.3506	0.3119	0.3164	0.3120	0.2463
0.7571	0.3558	0.3164	0.3204	0.3164	0.2476
0.8000	0.3599	0.3200	0.3235	0.3200	0.2485
0.8360	0.3630	0.3228	0.3260	0.3228	0.2490

The compaction parameter corresponding to homogeneous density distribution (Schwarzschild's interior solution) is indicated by u_h for the same value of σ. It is seen that for each and every assigned value of σ, u_{BDL}, u_{VII}, u_{V}, and $u_{\mathrm{VI}} < u_h$, which is the evidence that the regular solutions corresponding to a vanishing value of the surface density (represented by u_{BDL} and u_{VII} respectively), and singular solutions having a non-vanishing value of the surface density (represented by u_{V} and u_{VI} respectively) are compatible with the structure of general relativity.

"gaseous" solution, and the singular solutions with non-vanishing surface density, namely - Tolman's V and VI solutions are compatible with the structure of general relativity. The only regular solution with finite non-vanishing surface density which could exist in this regard is described by constant (homogeneous) density distribution.

This criterion provides a necessary and sufficient condition for any static spherical configuration to be compatible with the structure of general relativity,

and may be used to construct core-envelope models of stellar objects like neutron stars with vanishing and non-vanishing surface densities, such that for an assigned value of central pressure to central density, the average density of the configuration should always remain less than or equal to the density of the homogeneous sphere for the same mass.

This criterion could provide a convenient and reliable tool for testing equations of state (EOSs) for dense nuclear matter and models of relativistic star clusters, and may find application to investigate new analytic solutions and EOSs.

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